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On refinements of ω-bounded group topologies

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A topological group is ω-bounded if the closure of any countable subset is compact. Clearly, the ω-bounded groups are countably compact and hence, precompact. It has been pointed out recently that the class of ω-bounded groups is related with that of P-groups by duality (Galindo et al., 2011 [7]). In this direction, we obtain a characterization of ω-bounded topological groups by means of a property of the dual group (Theorem 2.4), and from it we deduce that a precompact group is realcompact if and only if its P-modification is complete (Theorem 3.5). Finally, we prove that for an ω-bounded group G, the next assertions are equivalent (Theorem 4.1):
a) There exists an ω-bounded group topology on G strictly finer than the original.
b) The dual group of G with the pointwise convergence topology is not realcompact.
c) The P-modification of the dual group with the pointwise convergence topology is not complete.
An important result of Comfort and van Mill establishes that for every pseudocompact Abelian topological group of uncountable weight (G, τ) there exists a pseudocompact group topology strictly finer than τ, in other words, τ is not r-extremal. In this paper we prove that the smaller class of ω-bounded groups behaves in a substantially different mode: namely, for an ω-bounded Abelian topological group there always exists a supreme ω-bounded group topology finer than the original one (Corollary 4.2). The latter plays thus the role of r-extremal in the class of ω-bounded group topologies.

1. Introduction and notation

The word duality appears in many branches of Mathematics. Loosely speaking a duality theory is an assignment between two categories of objects, reflecting some properties of the first category into properties of the other one, which might be called "dual properties".

In the framework of Abelian topological groups there is a formidable duality theory, introduced by Pontryagin in the 30s of the past century. Since then there has been considerable activity concerning this topic, but still one can find open problems. This paper aims to be a contribution to duality between two classes of topological groups: the ω-bounded and the P-groups.

Let us first recall that for a topological Abelian group G, the Pontryagin dual (or simply the dual) is the group of continuous homomorphisms, $G^\wedge := CHom(G, \mathbb{T})$, where $\mathbb{T}$ denotes the circle group. Precisely because of Pontryagin–van Kampen duality theorem, the natural topology for the dual group is the compact-open topology, and frequently in the Literature by "dual group" it is understood the dual already topologized with it. Obviously there are other important group topologies on $G^\wedge$: along this paper we will mainly be concerned with the pointwise convergence topology. In the sequel, $G^\wedge$ denotes the dual group of G, without any topology, and $G_c^\wedge$ the dual group with the compact-open topology.

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A topological group G is *ω-bounded* if the closure of every countable subset of G is compact or, equivalently, if the closure of every countable subgroup of G is compact. Clearly every ω-bounded topological group is pseudocompact. As proved by Comfort and Ross in [6], every pseudocompact group is precompact.

A *P-group* is a topological group in which every G_δ-set is open. Any topological group (G, τ) (briefly G_τ) gives rise canonically to a P-group $(G, P\tau)$ (or $G_{P\tau}$), where $P\tau$ is the group topology generated by all G_δ-sets in (G, τ). The topology $P\tau$ will be called the *P-modification* of τ. We also denote by PG the group G endowed with $P\tau$, where τ is the original topology of the group G. It is a well-known fact that a P-group G does not have infinite compact subsets, and hence the compact-open topology in $G^\wedge$ is the pointwise convergence topology $\sigma(G^\wedge, G)$. Further, if G is a P-group then $G_c^\wedge$ is ω-bounded ([2] or [7]).

In this article it is shown that a precompact Hausdorff Abelian group A is ω-bounded if and only if every $P\sigma(A^\wedge, A)$-continuous character defined on $A^\wedge$ is $\sigma(A^\wedge, A)$-continuous.

Many authors have dealt with the question of finding a strictly finer pseudocompact group topology for a pseudocompact group. In this line, a pseudocompact group G is called *r-extremal* if it does not admit a strictly finer pseudocompact group topology. Comfort and van Mill prove in [4] that the only r-extremal pseudocompact Abelian groups are those of countable weight, hence compact metrizable.

In this paper we study similar questions in the smaller class of ω-bounded groups. We prove that an ω-bounded Hausdorff Abelian group is r-extremal (in the class of ω-bounded groups) if and only if its dual group endowed with the pointwise convergence topology is realcompact. Further, for every ω-bounded Hausdorff Abelian group G there exists a finest ω-bounded group topology on G among all those group topologies finer than the original of G.

We also prove that a precompact Hausdorff Abelian group is realcompact if and only if its P-modification is Raikov complete. The realcompactness of precompact (or even ω-narrow) topological groups has been studied by several authors, see for example [1, Problems 5.1.D and 5.1.E], [11, Section 2.4] and [12]. In the class of precompact Abelian groups, realcompactness and pseudocompactness have recently been characterized in terms of properties of the dual group [9].

Notation. All the groups considered are Abelian, we omit this term in the sequel. We denote by $\mathbb{R}$ the additive group of real numbers endowed with the Euclidean topology, by $\mathbb{T}$ the quotient topological group $\mathbb{R}/\mathbb{Z}$, and let $\mathbb{T}_+ := [-1/4, 1/4]+\mathbb{Z} \subset \mathbb{T}$.

For a group G, $H \leqslant G$ means that H is a subgroup of G and $H < G$ means that H is a proper subgroup of G. For a subset $C \subset G$, we define $\langle C\rangle$ as the subgroup of G generated by C. If D and E are subgroups of the group G, then $D + E := \langle D \cup E\rangle$.

Let (G, τ) be a topological group, $H \leqslant (G, \tau)$ means that H is a subgroup of G endowed with $\tau\restriction_H$.

The symbol 0 denotes the neutral element of a group G and $G^* := Hom(G, \mathbb{T})$ denotes all the homomorphisms from G to $\mathbb{T}$, also called characters of G. It is clear that G^* is a group with respect to the pointwise operation. If G is a topological group, then $G^\wedge \leqslant G^*$.

Let H be a subgroup of G^* which separates the points of an Abelian group G. We denote by $\sigma(G, H)$ the weak topology induced by H on G. It is a group topology and $(G, \sigma(G, H))$ is a precompact Hausdorff topological group. In [5] it is proved that $(G, \sigma(G, H))^\wedge = H$, and that every precompact Hausdorff group G carries the topology induced by its continuous characters, namely $\sigma(G, G^\wedge)$. So, if G is a precompact Hausdorff group then $G^\wedge$ separates the points of G and G coincides with $(G, \sigma(G, G^\wedge))$. In the sequel we assume that all the precompact groups are Hausdorff, therefore if G is a precompact group then $G^\wedge$ separates the points of G.

For a precompact group G, the Bohr compactification of G is defined as $bG := (G^{\wedge *}, \sigma(G^{\wedge *}, G^\wedge))$, where $G^{\wedge *} = Hom(G^\wedge, \mathbb{T})$ is the group of all the homomorphisms from $G^\wedge$ to $\mathbb{T}$. Observe that bG is a compact topological group and G can be identified with a dense subgroup of bG by means of the canonical embedding of G into $G^{\wedge *}$. The latter assigns to each element $x \in G$ the character of $G^\wedge$ defined by evaluation at x, thus $G \leqslant bG$. It is clear that bG is simply the (Weil) completion ρG of the precompact group G.

The dual group of G endowed with the compact-open topology is a topological group denoted by $G_c^\wedge$. The symbol $\sigma(G^\wedge, G)$ stands for the topology on $G^\wedge$ defined by pointwise convergence on the elements of G (considered as characters on $G^\wedge$) and briefly we write $G_p^\wedge := (G^\wedge, \sigma(G^\wedge, G))$. If the topological group G does not contain any infinite compact subset, then $G_c^\wedge = G_p^\wedge$.

Let G be a topological group. For a subset $C \subset G$, define $C^{\triangleright G^*} := \{\chi \in G^*\colon\ \chi(C) \subset \mathbb{T}_+\}$, and the polar set of C as $C^\triangleright := C^{\triangleright G^*} \cap G^\wedge$. The annihilators of C are defined by $C^{\perp G^*} := \{\chi \in G^*\colon\ \chi(C) = \{0\}\}$ and $C^\perp := C^{\perp G^*} \cap G^\wedge$. It is easy to see that both $C^{\perp G^*}$ and $C^\perp$ are subgroups of G^* and $G^\wedge$ respectively. Further, if C is a subgroup of G, then $C^{\triangleright G^*} = C^{\perp G^*}$ and $C^\triangleright = C^\perp$.

Let $G^\wedge$ be the dual group of a topological group G, and let $D \subset G^\wedge$. The inverse polar of D is defined by $D^\triangleleft := D^{\triangleright G^{\wedge *}} \cap G = \{x \in G\colon\ \chi(x) \in \mathbb{T}_+$ for each $\chi \in D\}$, and the annihilators by $D^{\perp G^{\wedge *}} = D^{\perp bG} := \{f \in G^{\wedge *}\colon\ f(\chi) = 0$ for each $\chi \in D\}$ and by $D^\perp := D^{\perp G^{\wedge *}} \cap G$. In the definitions of $D^\triangleleft$ and $D^\perp$, the group G has been identified with the corresponding subgroup of $bG = G^{\wedge *}$. It is easy to check that $(\bigcup_{i\in I} D_i)^\triangleleft = \bigcap_{i\in I}(D_i)^\triangleleft$.

2. ω-Bounded groups and P-groups

In this section we relate the dual group of a precompact group G, with the dual of its P-modification PG. We rely on the fact that both duals are subgroups of G^*.

We first represent the dual group of a topological group G in terms of a local basis at the neutral element of G.

Lemma 2.1. *Let G be a topological group and let $\mathcal{B}$ be a basis of neighborhoods of 0. Then $G^\wedge = \bigcup_{U\in\mathcal{B}} U^{\triangleright G^*}$.*

Proof. Observe that for a neighborhood of 0, U, any character χ such that $\chi(U) \subset \mathbb{T}_+$ must be continuous. Therefore $U^{\triangleright G^*} = U^{\triangleright}$ and it is straightforward that $G^\wedge = \bigcup_{U\in\mathcal{B}} U^{\triangleright G^*}$. □

Consider now the P-modification of a precompact topology. In this particular case, the annihilators of the countable subgroups of the dual group form a basis of neighborhoods of zero for the P-modification, as we prove in the next lemma.

Lemma 2.2. *Let G be a precompact group. Then the family $\mathcal{B} = \{B^\perp\colon B \leqslant G^\wedge,\ B \text{ countable}\}$ is a local basis at the neutral element of the group PG.*

Proof. It is known that $\mathcal{B}_\sigma = \{F^\triangleleft\colon F \subset G^\wedge,\ F \text{ finite}\}$ is a basis of neighborhoods of zero in G. Then, for $B \leqslant G^\wedge$ with $|B| \leqslant \omega$, we have $B^\perp = B^\triangleleft = (\bigcup_{\chi\in B}\{\chi\})^\triangleleft = \bigcap_{\chi\in B}\{\chi\}^\triangleleft$, which is a countable intersection of neighborhoods of zero in G, therefore a neighborhood of zero in PG. Conversely, take a countable intersection of neighborhoods of zero in G, say $\bigcap_{i\in I}(F_i)^\triangleleft$, where F_i is finite for all $i \in I$ and $|I| \leqslant \omega$. Define the countable subgroup $B := \langle\bigcup_{i\in I} F_i\rangle$; clearly $B^\perp \subset \bigcap_{i\in I}(F_i)^\triangleleft$ and so $\mathcal{B}$ is a local basis at zero for the group PG, the P-modification of G. □

We now describe the dual group of the P-modification of a precompact group.

Theorem 2.3. *Let PG be the P-modification of a precompact group G. Then $A := (PG)^\wedge = \bigcup_{B\leqslant G^\wedge, |B|\leqslant\omega} \overline{B}^{G^*}$, where $\overline{B}^{G^*}$ is the closure of B in $(G^*, \sigma(G^*, G))$.*

Further, $(PG)^\wedge_c = (A, \sigma(A, G))$ is an ω-bounded group, which can be considered as the ω-bounded hull of $G^\wedge_p$ in the following sense: there does not exist a proper subgroup C with $G^\wedge \leqslant C < A$ such that $(C, \sigma(C, G))$ is ω-bounded.

Proof. By means of Lemmas 2.2 and 2.1 we obtain

$$(PG)^\wedge = \bigcup_{B\leqslant G^\wedge, |B|\leqslant\omega} \left(B^\perp\right)^{\triangleright G^*} = \bigcup_{B\leqslant G^\wedge, |B|\leqslant\omega} \left(B^\perp\right)^{\perp G^*} = \bigcup_{B\leqslant G^\wedge, |B|\leqslant\omega} \overline{B}^{G^*}. \tag{1}$$

Let us see now that if (G, τ) is a P-group, then its dual group with the compact-open topology is ω-bounded (this fact is also proved in [2] and [7], we give an alternative argument here).

As (G, τ) is a P-group, it has no infinite compact subsets, and hence $(G, \tau)^\wedge_c = (A, \sigma(A, G))$ is a precompact group. Clearly $\tau \supseteq \sigma(G, A)$, and this implies $\tau = P\tau \supseteq P\sigma(G, A) \supseteq \sigma(G, A)$. Then, we have $(G, P\sigma(G, A))^\wedge = (G, \tau)^\wedge = A$. By (1), $A = \bigcup_{B\leqslant A, |B|\leqslant\omega} \overline{B}^{G^*}$. Since $(G^*, \sigma(G^*, G))$ is compact, for each countable subgroup B of A, $\overline{B}^{G^*}$ is also compact, and it is inside A. Thus A is ω-bounded.

The definition of the group A makes it clear that there cannot exist any group C with $G^\wedge \leqslant C < A$ such that $(C, \sigma(C, G))$ is ω-bounded. □

A compact topological group is obviously ω-bounded and its dual group, being discrete, is a P-group. We characterize now ω-bounded topological groups by means of a property of their dual groups. Namely, the pointwise convergence topology in the dual of an ω-bounded group and its P-modification are compatible topologies (in the sense that they admit the same continuous characters).

Theorem 2.4. *Let G be a group and A a subgroup of G^*. Then the group $(A, \sigma(A, G))$ is ω-bounded if and only if $(G, P\sigma(G, A))^\wedge = A$, i.e., every $P\sigma(G, A)$-continuous character of G is $\sigma(G, A)$-continuous.*

Proof. Assume that $A_\sigma := (A, \sigma(A, G))$ is ω-bounded. Then for every countable subgroup $B \leqslant A_\sigma$ we have that $\overline{B}^{A_\sigma}$ is compact and, as A_σ is a topological subgroup of the compact Hausdorff group $(G^*, \sigma(G^*, G))$, we obtain $\overline{B}^{A_\sigma} = \overline{B}^{G^*}$. This, together with Theorem 2.3 (applied to the precompact group $(G, \sigma(G, A))$) implies

$$\left(G, P\sigma(G, A)\right)^\wedge = \bigcup_{B\leqslant A, |B|\leqslant\omega} \overline{B}^{G^*} = \bigcup_{B\leqslant A, |B|\leqslant\omega} \overline{B}^{A_\sigma} = A.$$

The converse follows from Theorem 2.3. □

Remark 2.5. In [10] a subclass of precompact groups is introduced under the name ULQC (unique locally quasi-convex compatible). A precompact group (G, τ) is in the class ULQC if for any locally quasi-convex group topology ν on G, the equality $(G, \nu)^\wedge = (G, \tau)^\wedge$ implies $\nu = \tau$. In other words, τ is the only locally quasi-convex group topology on G with dual group $G^\wedge$.

It is proved in [10, Theorem 8.44] that every ω-bounded group is g-barrelled, and in particular belongs to ULQC. Now Theorem 2.4 proves that the class ULQC is not autodual in the following sense: the dual $G^\wedge$ of an ω-bounded group G endowed again with its weak topology $\sigma(G^\wedge, G)$ may not be in ULQC. Clearly $\sigma(G^\wedge, G)$ and $P\sigma(G^\wedge, G)$ are in general two distinct locally quasi-convex compatible topologies. The g-barrelled groups were introduced in [3].

3. Completeness in P-groups and realcompactness

The notion of sequentially continuous homomorphism can be restricted to obtain a new type of characters, called countably continuous characters. They constitute an important tool in order to characterize realcompactness, as well as pseudocompactness and related facts. We recall now the definition of a countably continuous character of a topological group G, and we obtain an expression for the group of $\sigma(G^\wedge, G)$-countably continuous characters of the dual of a precompact group G.

Definition 3.1. Let (G, τ) be a topological group. A character $\chi \in G^*$ is said to be τ-countably continuous if $\chi\restriction_B$ is $\tau\restriction_B$-continuous for every countable subset $B \subset (G, \tau)$. Equivalently, if $\chi\restriction_B$ is $\tau\restriction_B$-continuous for every countable subgroup $B \leqslant (G, \tau)$.

Proposition 3.2. *Let G be a precompact group. Then the group of $\sigma(G^\wedge, G)$-countably continuous homomorphisms from $G^\wedge$ to $\mathbb{T}$ is $D = \bigcap_{B \leqslant G^\wedge, |B| \leqslant \omega}(G + B^{\perp G^{\wedge *}})$. Clearly $G \leqslant D \leqslant G^{\wedge *}$.*

Proof. Let $B \leqslant (G^\wedge, \sigma(G^\wedge, G))$ be a countable subgroup. Since B is a subgroup of a precompact group, every $\varphi \in B^\wedge$ can be extended to a continuous character defined on $(G^\wedge, \sigma(G^\wedge, G))$, and therefore identified with an element of G. Thus, taking into account that $B^{\perp G} = B^{\perp bG} \cap G$, we have:

$$B^\wedge = G/B^{\perp G} = \left(G + B^{\perp bG}\right)/B^{\perp bG}.$$

From the last equality it can be deduced that the natural map $G + B^{\perp bG} \to B^\wedge$ is onto and hence,

$$G + B^{\perp bG} = \left\{f \in bG\colon f\restriction_B \text{ is } \sigma\left(G^\wedge, G\right)\restriction_B \text{-continuous}\right\}.$$

As $bG = G^{\wedge *}$, we obtain that

$$D = \bigcap_{B \leqslant G^\wedge, |B| \leqslant \omega} \left(G + B^{\perp G^{\wedge *}}\right) = \left\{f \in G^{\wedge *}\colon f \text{ is } \sigma\left(G^\wedge, G\right)\text{-countably continuous}\right\}. \qquad \square$$

In order to present completeness properties of the group D described in Proposition 3.2, we state the following auxiliary result.

Lemma 3.3. *([8, Theorem 8]) The P-modification of a complete topological (not necessarily Abelian) group is complete.*

For the next theorem, we recall that the Raikov completion of a topological group G is a complete topological group ρG which densely contains G. Further ρG is essentially unique.

Theorem 3.4. *Let PG be the P-modification of a precompact group G. Then the Raikov completion of PG, $\rho(PG)$, is the topological group $D_{P\sigma} = (D, P\sigma(D, G^\wedge))$, where D is the group defined in Proposition 3.2.*

Proof. By Lemma 3.3, the P-modification of a compact group is complete, hence $P(bG) = (G^{\wedge *}, P\sigma(G^{\wedge *}, G^\wedge))$ is complete. Since $PG \leqslant P(bG)$ we can consider $\rho(PG)$ as the closure of G in $P(bG)$. By Lemma 2.2, $\mathcal{B} = \{B^{\perp bG}\colon B \leqslant G^\wedge,\ B \text{ countable}\}$ is a local basis at zero of the topological group $P(bG)$, thus:

$$\rho(PG) = \overline{G}^{P(bG)} = \bigcap_{U \in \mathcal{B}}(G + U) = \bigcap_{B \leqslant G^\wedge, |B| \leqslant \omega} \left(G + B^{\perp G^{\wedge *}}\right) = D_{P\sigma}. \qquad \square$$

For a given precompact group G, we now relate the properties of being realcompact with the completeness of its P-modification and with the group D of $\sigma(G^\wedge, G)$-countably continuous characters.

Theorem 3.5. *The following conditions are equivalent for a precompact Abelian group G:*

a) *G is realcompact.*
b) *Every $\sigma(G^\wedge, G)$-countably continuous homomorphism $f \in G^{\wedge *}$ is $\sigma(G^\wedge, G)$-continuous, i.e., $D = G$.*
c) *The topological group PG is complete.*

Proof. a) ⇔ c): It is known that the precompact group G is realcompact if and only if it is G_δ-closed in its Bohr compactification $bG = (G^{\wedge *}, \sigma(G^{\wedge *}, G^{\wedge}))$ (for example see [13, Exercise 1C]). This is equivalent to the statement that G is closed in $P(bG)$, and equivalently PG is complete (Lemma 3.3).

b) ⇔ c): By Theorem 3.4, PG is complete if and only if $G = D$, where D is the group of $\sigma(G^{\wedge}, G)$-countably continuous homomorphisms. □

Remark 3.6. In [9] it was proved that a) and b) are equivalent, our proof is different. On the other hand, the equivalence of items a) and c) can also be deduced from [1, Corollary 8.1.17 and Theorem 8.3.6].

Remark 3.7. In [9] it is also proved that the following assertions are equivalent for a precompact group G:

i) The group G is pseudocompact.
ii) $B^{\wedge} = B^*$ for every countable $B \leqslant (G^{\wedge}, \sigma(G^{\wedge}, G))$.
iii) Every $f \in G^{\wedge *}$ is $\sigma(G^{\wedge}, G)$-countably continuous.

By Theorem 3.4 it is clear that these assertions are equivalent to the equality $\rho(PG) = P(bG)$.

Theorem 3.5 and Remark 3.7 together stress the fact that realcompactness and pseudocompactness are opposite notions. In fact, the precompact group G is realcompact if the group D of countably continuous characters of $G_p^{\wedge}$ is as small as possible, namely it coincides with G. Whilst G is pseudocompact if and only if D is as big as is possible, every character defined on $G^{\wedge}$ is countably continuous, i.e. $D = bG$.

Corollary 3.8. *If a precompact group G is not realcompact, then the group $D_\sigma = (D, \sigma(D, G^{\wedge}))$ of $\sigma(G^{\wedge}, G)$-countably continuous homomorphisms is a realcompactification of G and no topological group C with $G \leqslant C < D_\sigma$ is realcompact.*

Proof. By Theorem 3.4 $D_{P\sigma}$ is complete. Thus, by Theorem 3.5 D_σ is realcompact and since $G \leqslant D_\sigma \leqslant bG$, G is dense in D_σ.

For the second part assume that there exists a realcompact group C satisfying $G \leqslant C < D_\sigma$. By Theorem 3.5 PC is complete and now by Theorem 3.4 we obtain that $PC = P(D_\sigma) = D_{P\sigma}$, therefore $C = D_\sigma$. □

Corollary 3.9. *Suppose that G is a precompact realcompact group and let $A = (PG)^{\wedge}$. Then the group $(G, \sigma(G, H))$ is realcompact for every group H with $G^{\wedge} \leqslant H \leqslant A$.*

Proof. By Theorem 3.5 $(G, P\sigma(G, G^{\wedge}))$ is complete, but it is clear that $P\sigma(G, G^{\wedge}) = P\sigma(G, H)$ for every H satisfying the above condition. Again by Theorem 3.5, $(G, \sigma(G, H))$ is a realcompact group. □

4. Supreme ω-bounded topology

The following question has been intensively studied by several authors.

If (G, τ) is a pseudocompact Abelian group, is there a strictly finer pseudocompact group topology for G? Recently Comfort and van Mill solved the question. They proved in [4] that pseudocompact Abelian groups of uncountable weight admit a strictly finer pseudocompact group topology. The class of ω-bounded groups behaves in a substantially different way, as proved in Corollary 4.2. Namely, for an ω-bounded topological group (G, τ) the family of all ω-bounded group topologies on G finer than τ has a maximum.

Theorem 4.1. *Let G be a group and A a point-separating subgroup of G^* such that $(A, \sigma(A, G))$ is an ω-bounded topological group. The following assertions are equivalent:*

a) *There exists a topology $\tau \supset \sigma(A, G)$ (strictly finer) such that (A, τ) is an ω-bounded topological group.*
b) *The P-group $G_{P\sigma} = (G, P\sigma(G, A))$ is not complete.*
c) *The precompact group $(G, \sigma(G, A))$ is not realcompact.*

Proof. a) is fulfilled if and only if there exists a subgroup H, $G < H \leqslant A^*$, such that $(A, \sigma(A, H))$ is ω-bounded. Set $H_{P\sigma} := (H, P\sigma(H, A))$. By Theorem 2.4, $(A, \sigma(A, H))$ is ω-bounded if and only if $(H_{P\sigma})^{\wedge} = A$. On the other hand, Theorem 2.4 also implies $(G_{P\sigma})^{\wedge} = A$. As $G_{P\sigma} \leqslant H_{P\sigma}$ and $(G_{P\sigma})^{\wedge} = (H_{P\sigma})^{\wedge}$ we obtain that a) is equivalent to the claim that $G_{P\sigma}$ is dense in $H_{P\sigma}$ and $G \neq H$. For this, by Theorem 3.4, a necessary and sufficient condition is that $G \neq D$ (where D is the group defined in Proposition 3.2) or equivalently, $G_{P\sigma}$ is not complete.

The equivalence between b) and c) is already proved in Theorem 3.5. □

From Theorem 4.1 we obtain precisely the r-extremal ω-bounded groups:

Corollary 4.2. *Let G be a group and A a subgroup of G^* such that $(A, \sigma(A, G))$ is an ω-bounded topological group, and let D be the group of $\sigma(A, G)$-countably continuous homomorphisms. Then $\sigma(A, D)$ is an ω-bounded topology finer than $\sigma(A, G)$ and there are no ω-bounded group topologies strictly finer than $\sigma(A, D)$. In other words, $\sigma(A, D)$ is the supremum ω-bounded topology among all those finer than $\sigma(A, G)$.*

Example 4.3. Let us introduce the topological group $G = (\mathbb{Z}_2^{(I)}, \sigma(\mathbb{Z}_2^{(I)}, \mathbb{Z}_2^{(I)}))$, where I is an index set with $|I| > \omega$ and $\mathbb{Z}_2^{(I)}$ is the direct sum of I copies of the cyclic group $\mathbb{Z}_2$ of order 2. Clearly $bG = (\mathbb{Z}_2^I, \sigma(\mathbb{Z}_2^I, \mathbb{Z}_2^{(I)}))$, where $\mathbb{Z}_2^I$ is the product of I copies of $\mathbb{Z}_2$. For the ω-bounded group $A := (PG)_c^\wedge$ the following statements hold:

a) $A = \Sigma\mathbb{Z}_2$, where $\Sigma\mathbb{Z}_2 = \{\chi \in \mathbb{Z}_2^I\colon |\operatorname{supp}\chi| \leqslant \omega\}$ is the Σ-product of I copies of $\mathbb{Z}_2$.
b) $\sigma(A, G)$ is a supremum ω-bounded group topology.
c) $(\mathbb{Z}_2^{(I)}, \sigma(\mathbb{Z}_2^{(I)}, H))$, where $\mathbb{Z}_2^{(I)} \leqslant H \leqslant \Sigma\mathbb{Z}_2$, is a realcompact group.

Proof. a) Let C be a countable subgroup of $G^\wedge = \mathbb{Z}_2^{(I)}$. Clearly, there exists a countable subgroup B such that $C \leqslant B = \mathbb{Z}_2^{(J)} \times \{0\}^{I\setminus J} \leqslant G^\wedge$, where $J \subset I$ and $|J| \leqslant \omega$. Therefore $\overline{B}^{G^*}$ is the compact group $\mathbb{Z}_2^J \times \{0\}^{I\setminus J}$. By Theorem 2.3 we obtain that $A = \Sigma\mathbb{Z}_2$.

b) By Theorem 4.1 we have to prove that PG is complete, or equivalently that $D = G$ (Theorem 3.5). We recall that $D = \bigcap_{B\leqslant G^\wedge, |B|\leqslant\omega}(G + B^{\perp bG})$. As we have seen in a) we can assume that $B = \mathbb{Z}_2^{(J)} \times \{0\}^{I\setminus J}$ with $|J| \leqslant \omega$. Hence $B^{\perp bG} = \{0\}^J \times \mathbb{Z}_2^{I\setminus J}$ and therefore $G + B^{\perp bG} = \mathbb{Z}_2^{(J)} \times \mathbb{Z}_2^{I\setminus J}$. Now it is easy to check that $D = G$.

c) This follows from Corollary 3.9. □

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